Spatial Data II: Spatial Econometrics

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Applied Quantitative Methods II

MA in Social Sciences, Spring 2026

Today's goals

- Understand why spatial autocorrelation violates OLS assumptions
- Diagnose spatial dependence: Moran's I on residuals and LM tests
- Estimate the Spatial Error Model (SEM) and interpret λ
- Estimate the Spatial Lag Model (SLM/SAR) and interpret ρ
- Compute direct and indirect effects after SLM estimation
- Understand estimation methods: ML, IV/2SLS, and why OLS fails
- Choose between SEM, SLM, SDM, and extensions (SAC, SDEM)
- Compare models: LR tests for nested models, AIC for non-nested

Roadmap

A standard OLS regression

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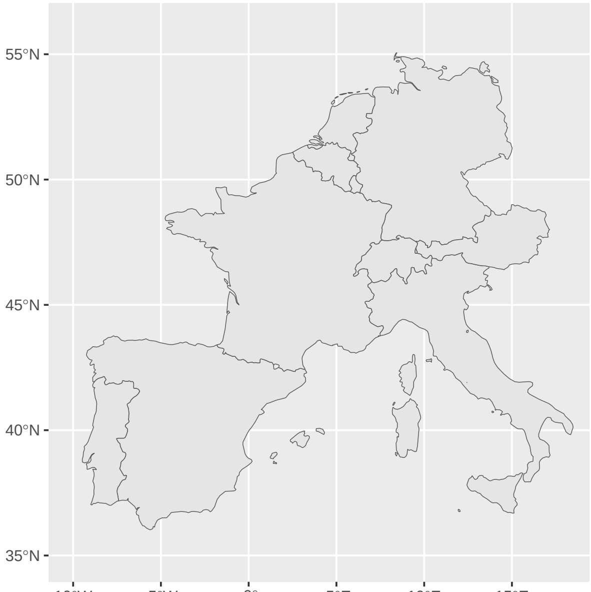
  - → But the structure of dependence is explicitly *geographic*

Spatial weights: a brief recap



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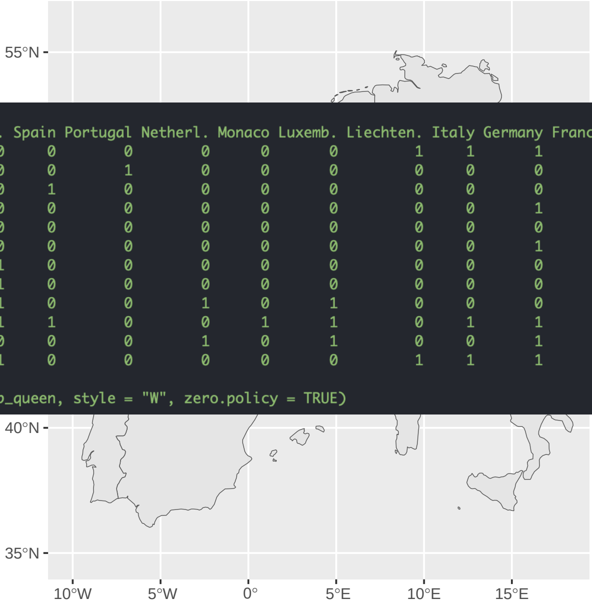


```
> mat
              Switzer. Spain Portugal Netherl. Monaco Luxemb. Liechten. Italy Germany France Belgium Austria
Switzerland          0     0        0        0      0       0         1     1       1      1       0       1
Spain                0     0        1        0      0       0         0     0       0      1       0       0
Portugal             0     1        0        0      0       0         0     0       0      0       0       0
Netherlands          0     0        0        0      0       0         0     0       1      0       1       0
Monaco               0     0        0        0      0       0         0     0       0      1       0       0
Luxembourg           0     0        0        0      0       0         0     0       1      1       1       0
Liechtenstein        1     0        0        0      0       0         0     0       0      0       0       1
Italy                1     0        0        0      0       0         0     0       0      1       0       1
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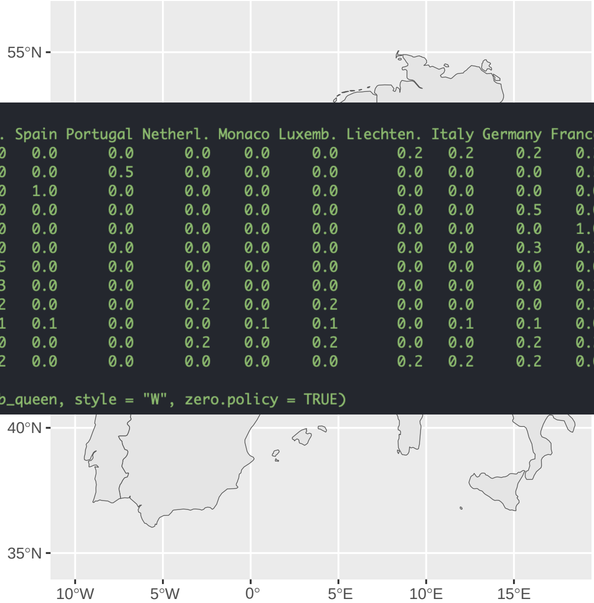


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Spain              0.0   0.0      0.5      0.0    0.0     0.0       0.0   0.0     0.0    0.5     0.0     0.0
Portugal           0.0   1.0      0.0      0.0    0.0     0.0       0.0   0.0     0.0    0.0     0.0     0.0
Netherlands        0.0   0.0      0.0      0.0    0.0     0.0       0.0   0.0     0.5    0.0     0.5     0.0
Monaco             0.0   0.0      0.0      0.0    0.0     0.0       0.0   0.0     0.0    1.0     0.0     0.0
Luxembourg         0.0   0.0      0.0      0.0    0.0     0.0       0.0   0.0     0.3    0.3     0.3     0.0
Liechtenstein      0.5   0.0      0.0      0.0    0.0     0.0       0.0   0.0     0.0    0.0     0.0     0.5
Italy              0.3   0.0      0.0      0.0    0.0     0.0       0.0   0.0     0.0    0.3     0.0     0.3
Germany            0.2   0.0      0.0      0.2    0.0     0.2       0.0   0.0     0.0    0.2     0.2     0.2
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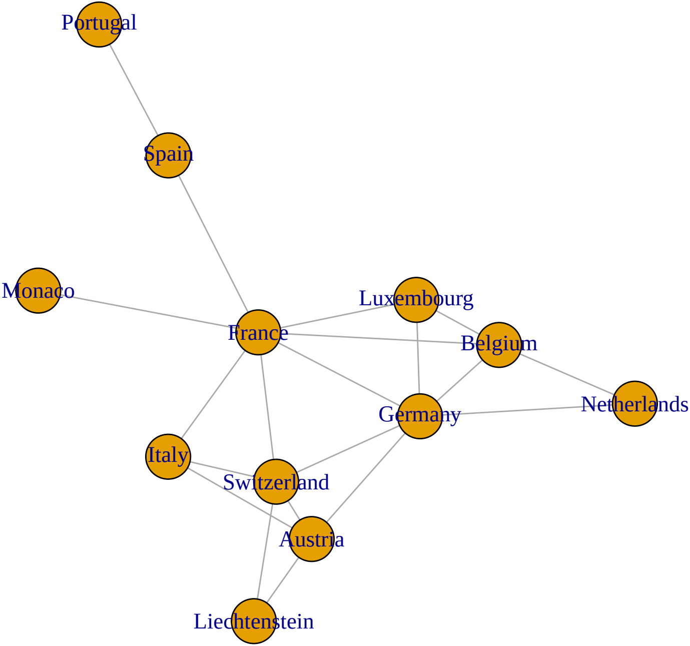
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  (but you could use **the same models on network data**!)

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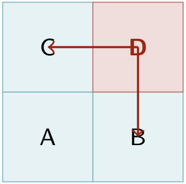

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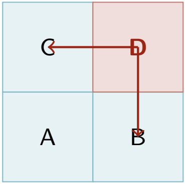

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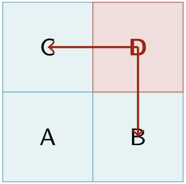

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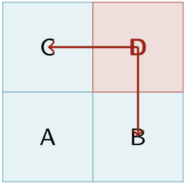

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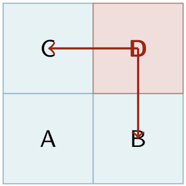

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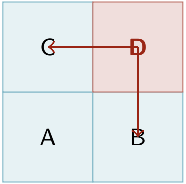

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- In R: `poly2nb()` → `nb2listw()`

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Two sources of spatial dependence

Moran's I on OLS residuals is significant

— why?

Nuisance correlation
Unmeasured spatially correlated factors

True spillover
Outcome in unit i affects neighbors j

⇒ Spatial Error Model

⇒ Spatial Lag Model

Same symptom, different model

Think about **your projects**: which mechanism is more plausible?

LM tests: which model to choose?

- **LM tests** (Lagrange Multiplier): test for specific types of spatial dependence

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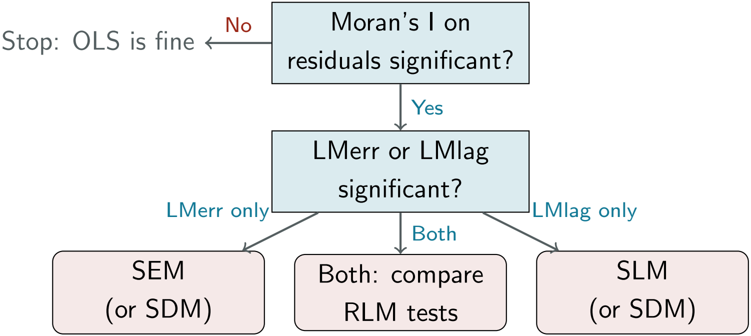
LM decision rule



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- The spatial structure is in the **residuals**, not in the outcome

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- $\lambda \neq 0$ does **not** mean y affects neighbors' y
  - → It means the residuals are spatially structured
- Fix: correct SEs and improve efficiency — not model substantive spillover

SEM in R

```
library(spatialreg) # or spdep

# Fit Spatial Error Model
sem_fit = errorsarlm(y ~ x1 + x2,
    data = mydata,
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summary(sem_fit)
```

- `errorsarlm()`: maximum likelihood estimation of SEM

SEM in R

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library(spatialreg) # or spdep

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- `errorsarlm()`: maximum likelihood estimation of SEM
- `zero.policy = TRUE`: allow units with no neighbors
- Key output: $\hat{\lambda}$ and its significance; $\hat{\beta}$ interpreted as OLS

SEM: reading the output

	Estimate	**Std. Error**	***z*-value**	***p*-value**
(Intercept)	-2.14	0.48	-4.45	< 0.001
`gdp_pc`	0.63	0.07	9.11	< 0.001
`pop_l`	0.18	0.05	3.37	0.001
λ (Lambda)	0.54	0.09	5.80	< 0.001

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- Use AIC/LR test to confirm SEM improves on OLS

Roadmap

SLM: the model

Spatial Lag Model (SLM / SAR)

$$\mathbf{y} = \rho \mathbf{W}\mathbf{y} + \mathbf{X}\beta + \varepsilon, \quad \varepsilon \sim \mathcal{N}(\mathbf{0}, \sigma^2 \mathbf{I})$$

- ρ (rho): spatial autoregressive parameter on the **lagged outcome**

SLM: the model

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- y_i is partly determined by its neighbors' y_j

SLM: substantive interpretation

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- Warning: $\hat{\beta}$ from SLM is **not** the marginal effect of $\mathbf{x}$ on y

SLM in R

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# Fit Spatial Lag Model
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- Key output: $\hat{\rho}$ and its significance; $\hat{\beta}$ (not marginal effects!)
- Do not interpret $\hat{\beta}$ directly — use `impacts()` instead

SLM: reading the output

	Estimate	Std. Error	z-value	p-value
(Intercept)	-1.87	0.52	-3.59	< 0.001
gdp_pc	0.51	0.08	6.29	< 0.001
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Roadmap

Introduction to spatial dependence

The Spatial Error Model (SEM)

The Spatial Lag Model (SLM/SAR)

Direct and Indirect Effects

The Spatial Durbin Model (SDM)

Estimation Methods

Model Selection and Practical Guidance

Model Relationships and Extensions

Beyond Cross-Sections: Spatial Panel Models

Changing x_i creates a ripple through the network.

The coefficient $\hat{\beta}$ is not the marginal effect.

Why $\hat{\boldsymbol{\beta}}$ is not the marginal effect

Solving the SLM for $\mathbf{y}$:

$$\mathbf{y} = \rho \mathbf{W}\mathbf{y} + \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$$
$$(\mathbf{I} - \rho \mathbf{W})\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$$
$$\mathbf{y} = \underbrace{(\mathbf{I} - \rho \mathbf{W})^{-1}}_{\mathbf{B}}(\mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon})$$

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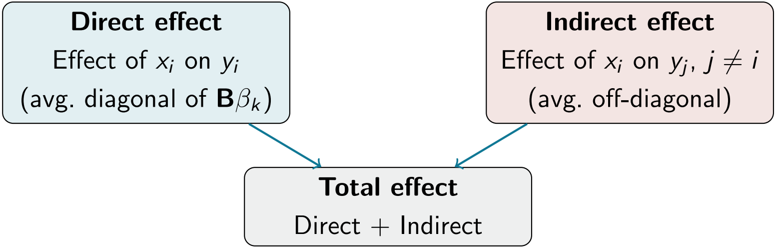
- **Question:** if you change x_i, how many units in the network are eventually affected?

Direct, indirect, and total effects



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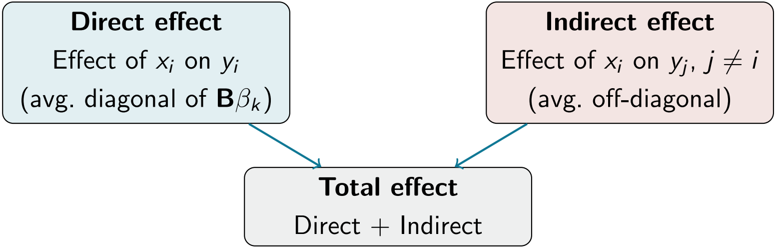

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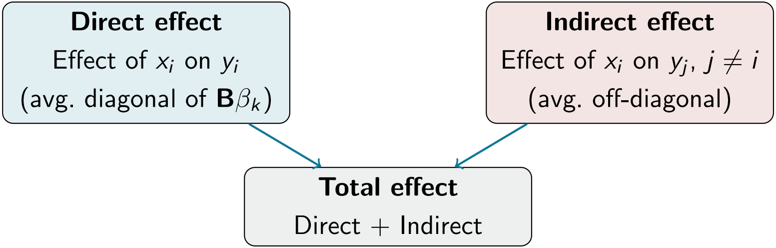

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- Indirect: captures spatial spillover — ignored by OLS and SEM
- Total: the full equilibrium effect of a uniform shock to x_k

Computing impacts in R

```
# Direct, indirect, total effects
imp = impacts(slm_fit, listw = my_listw)
summary(imp, zstats = TRUE)
```

```
## Impact measures (lag, exact):
##  Direct Indirect Total
# gdp_pc 0.574  0.531  1.105
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- **Direct** (gdp_pc): 0.574 — unit's own GDP raises own y by 0.574

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- Use `zstats = TRUE` to get z-statistics and p-values

Roadmap

SDM: extending the SLM

Spatial Durbin Model

$$\mathbf{y} = \rho \mathbf{W}\mathbf{y} + \mathbf{X}\boldsymbol{\beta} + \mathbf{W}\mathbf{X}\boldsymbol{\theta} + \boldsymbol{\varepsilon}$$

- Adds spatially lagged covariates $\mathbf{WX}\boldsymbol{\theta}$ to the SLM

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- LeSage & Pace (2009): SDM is often preferred as a conservative default

SDM in R and when to use it

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# Spatial Durbin Model
sdm_fit = lagsarlm(y ~ x,
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# Direct, indirect, total as usual
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- `Durbin = TRUE`: adds **WX** to `lagsarlm()`

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- Also useful as robustness check for SLM estimates

Roadmap

How are these models estimated?

Model	**Estimation**	**Why not OLS?**
OLS	OLS	Baseline: valid if no spatial dependence
SEM	ML	OLS is unbiased but **inefficient**; SEs are wrong
SLM	ML (or IV/2SLS)	$\mathbf{Wy}$ is **endogenous**: OLS is biased
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- Alternative for SEM: **Feasible GLS** or **GMM** (Kelejian & Prucha)

Roadmap

SEM vs SLM vs SDM: comparison

	SEM	**SLM**	**SDM**
What clusters	Errors ($\mathbf{u}$)	Outcome ($\mathbf{y}$)	Both
Spillover in y	No	Yes	Yes
Spillover from x	No	No	Yes
$\hat{\beta}$ **= marginal effect**	Yes	No	No
Use `impacts()`	No	Yes	Yes
Nuisance / design	Nuisance	Substantive	Both
R function	`errorsarlm()`	`lagsarlm()`	`lagsarlm( Durbin=TRUE)`

Choosing the right model: a guide

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  - → Is there genuine *diffusion* of the outcome across space? → **SLM / SDM**

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  - → Is spatial dependence a *nuisance* from omitted variables? → **SEM**
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Choosing the right model: a guide

- **Statistical guide** (LM tests):
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- **Robustness**: estimate all three, report side by side

The full workflow in R

```
library(sf); library(spdep); library(spatialreg)

# 1.  Build weights
nb = poly2nb(data, queen = TRUE)
listw = nb2listw(nb, style = "W", zero.policy = TRUE)

# 2.  OLS + diagnostics
ols = lm(y ~ x1 + x2, data = data)
moran.test(residuals(ols), listw = listw)
lm.LMtests(ols, listw, test = c("LMerr","LMlag",
     "RLMerr","RLMlag"))

# 3.  Spatial models
sem = errorsarlm(y ~ x1 + x2, data, listw)
slm = lagsarlm(y ~ x1 + x2, data, listw)
impacts(slm, listw = listw)
```

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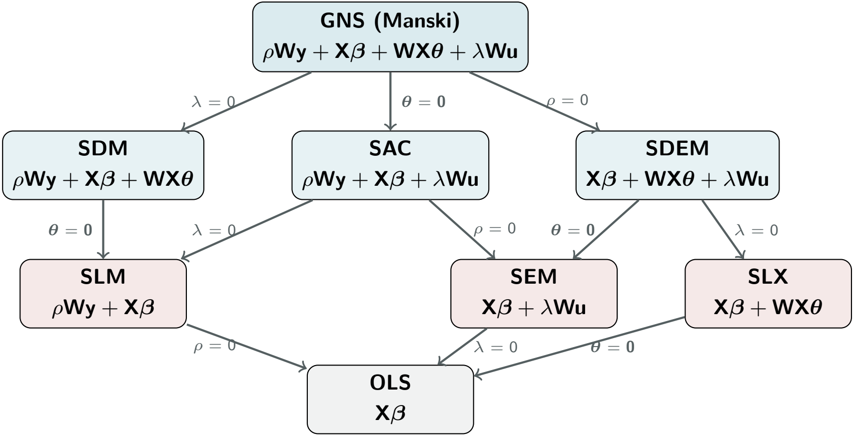
Roadmap

The family of spatial regression models



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- **Strategy**: use LR tests along nesting paths; use AIC across non-nested alternatives

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Spatial Autoregressive Combined Model

$$\mathbf{y} = \rho \mathbf{W}\mathbf{y} + \mathbf{X}\boldsymbol{\beta} + \mathbf{u}, \quad \mathbf{u} = \lambda \mathbf{W}\mathbf{u} + \varepsilon$$

- Combines spatial lag (ρ) and spatial error (λ) in one model

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sac_fit = sacsarlm(y ~ x1 + x2,
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- Neighbors' covariates ($\mathbf{W}\mathbf{X}\theta$) + spatially correlated errors (λ)

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Summary: nesting and comparison strategy

Comparison	Nested?	Test
OLS vs. SEM	Yes	LR test ($\lambda = 0$)
OLS vs. SLM	Yes	LR test ($\rho = 0$)
SLM vs. SDM	Yes	LR test ($\boldsymbol{\theta} = \mathbf{0}$)
SEM vs. SDM	Yes	LR test ($\boldsymbol{\theta} = -\rho\boldsymbol{\beta}$)
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- Key reference: Elhorst (2014), *Spatial Econometrics: From Cross-Sectional Data to Spatial Panels*
- **Bottom line**: if your data has a spatial *and* temporal dimension, use purpose-built tools — do not improvise with cross-sectional spatial models

For the lab session

- Complete the diagnostics workflow on a provided dataset
  - → Run OLS; compute Moran's I on residuals
  - → Run LM tests; select the appropriate model
- Estimate SEM and SLM; compare with OLS
  - → Interpret $\hat{\lambda}$ and $\hat{\rho}$
  - → Use `impacts()` for SLM marginal effects
- Reflect: what mechanism drives spatial dependence in your data?

Questions?